Properties of Knots from Polynomials

Ana Wright

April 13, 2023

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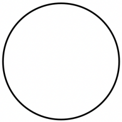

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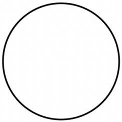

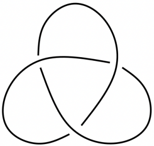

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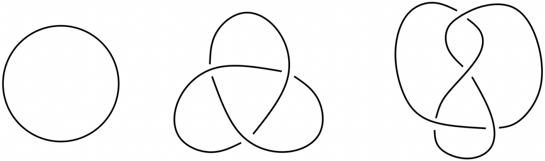

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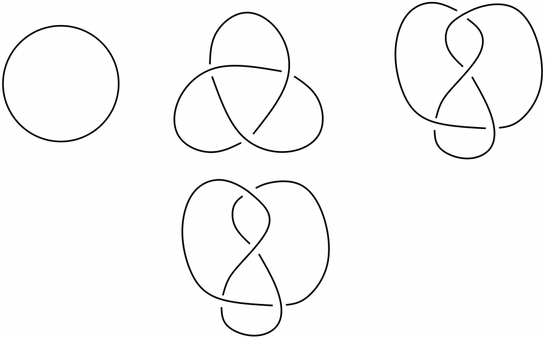

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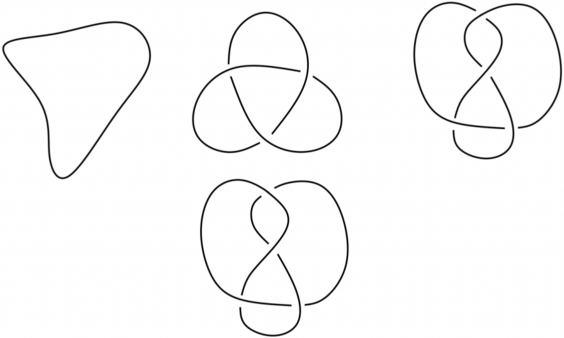

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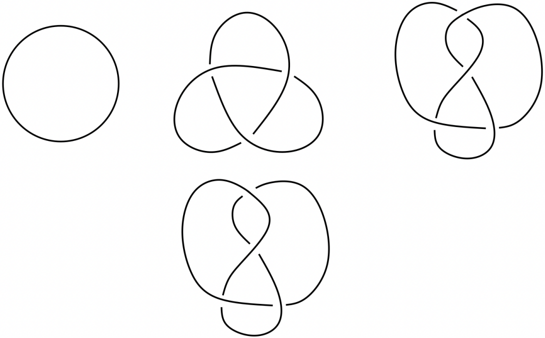

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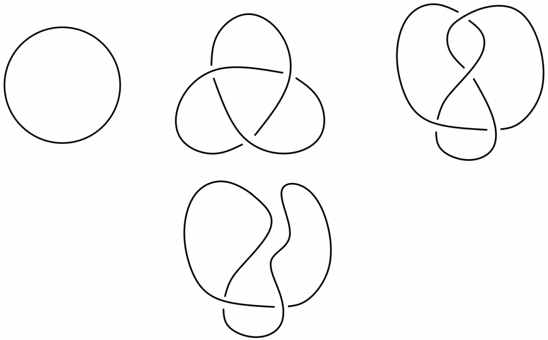

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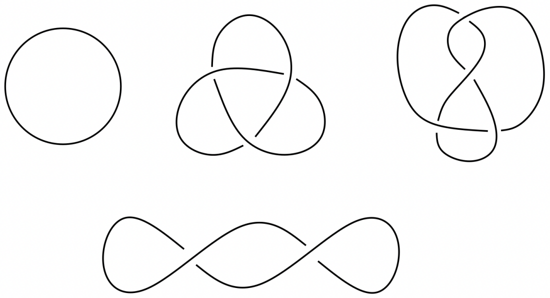

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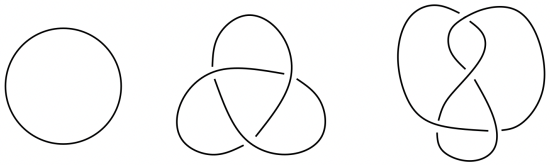

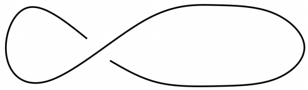

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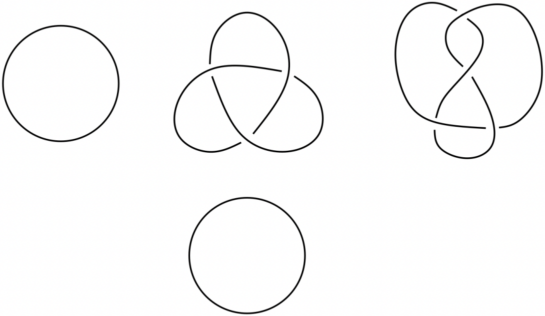

Reidemeister Moves (*R*-moves)

Theorem (Reidemeister, 1927. Independently, Alexander and Briggs, 1926)

Two knot diagrams are of the same knot if and only if one diagram can be transformed into the other through a series of the following Reidemeister moves and planar isotopies.

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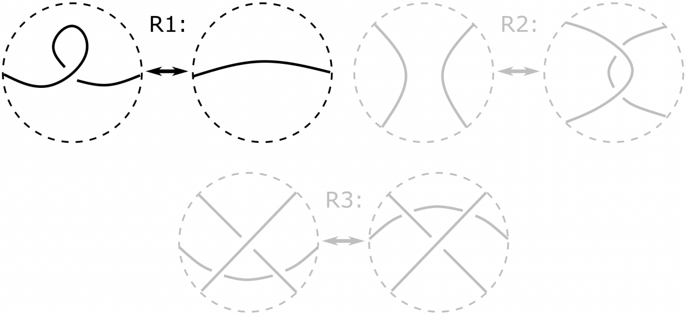

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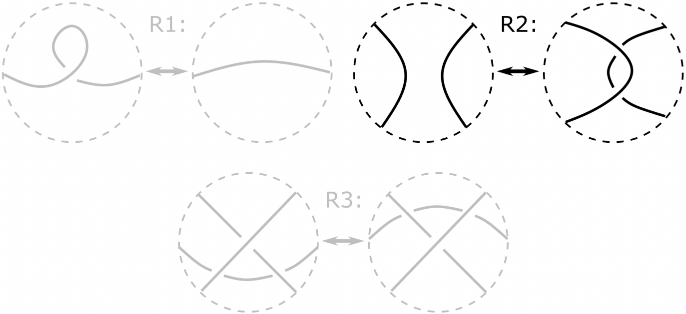

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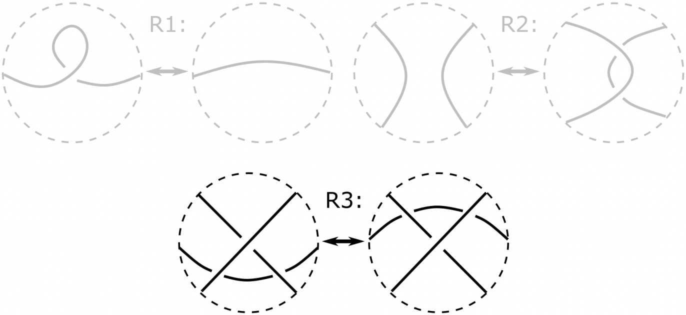

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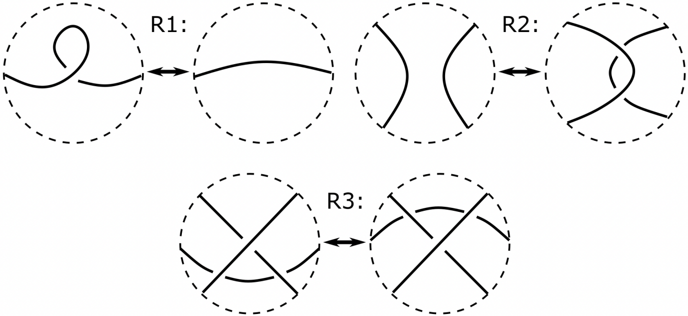


Knot Invariant

Definition

A **knot invariant** is a function f from the set of all knots $\mathcal{K}$ to a set S such that if two knots K and K' are equivalent, then $f(K) = f(K')$.

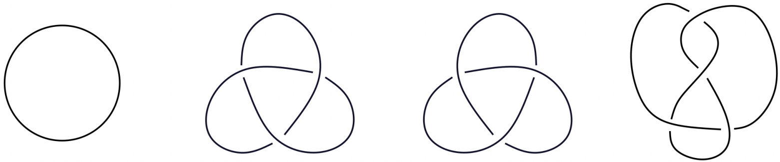
Type I: A property of knot diagrams which is invariant over continuous deformations.

Type II: A “measurement” which is minimized over all diagrams of a knot.

Type I Invariant Example

Definition

A knot diagram is **tricolorable** if the strands can be colored using exactly three colors such that every crossing uses either the same color for all three strands or all different colors.



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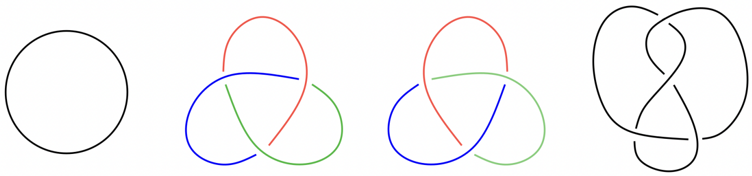

Tricolorability

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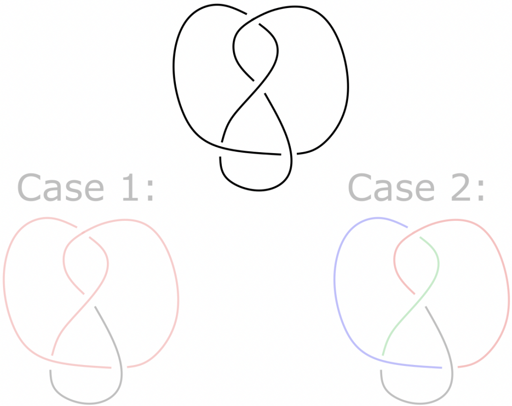

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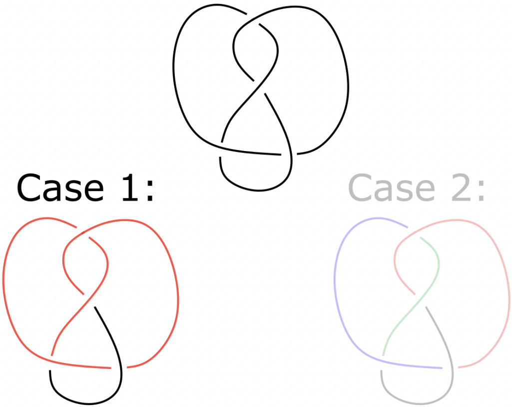

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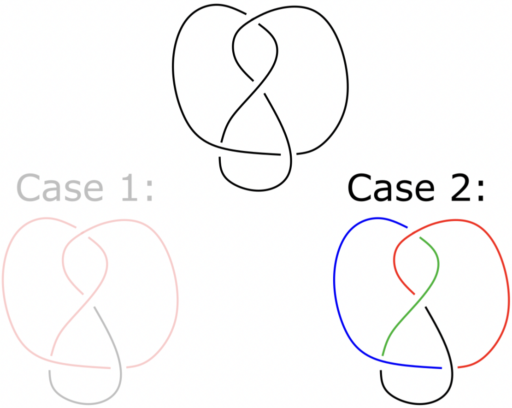

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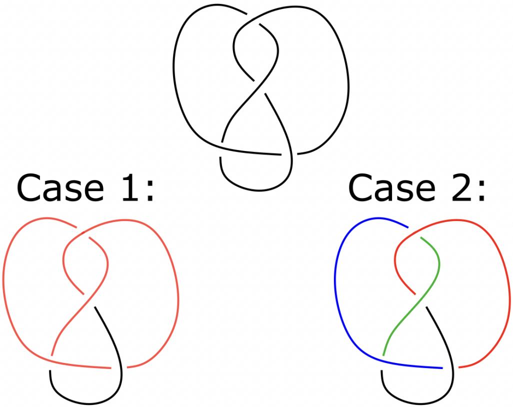
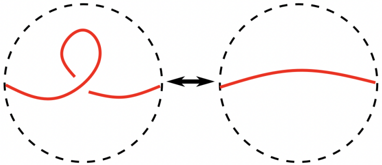


Tricolorability: Knot Invariant

Tricolorability is preserved by R-moves.



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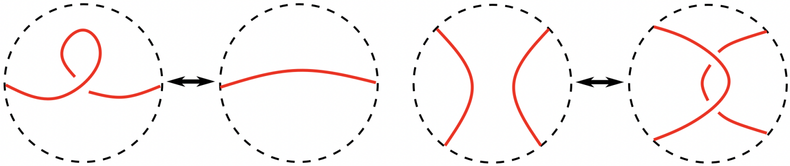

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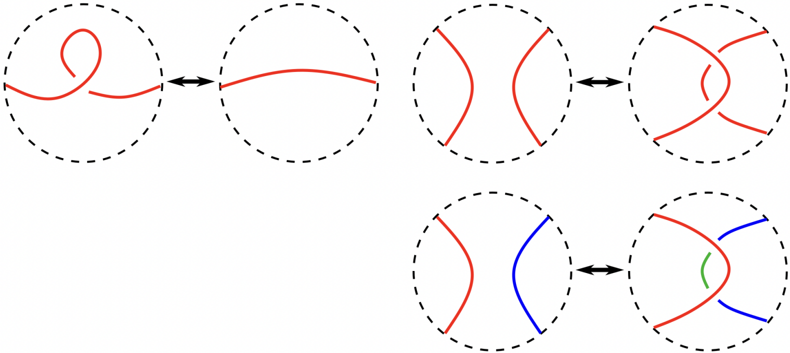

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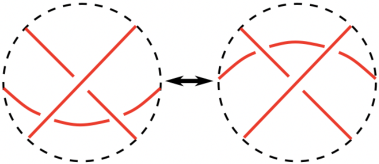

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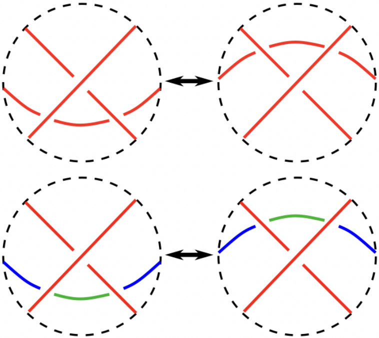

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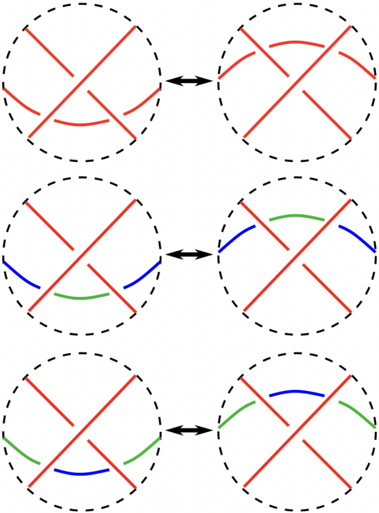

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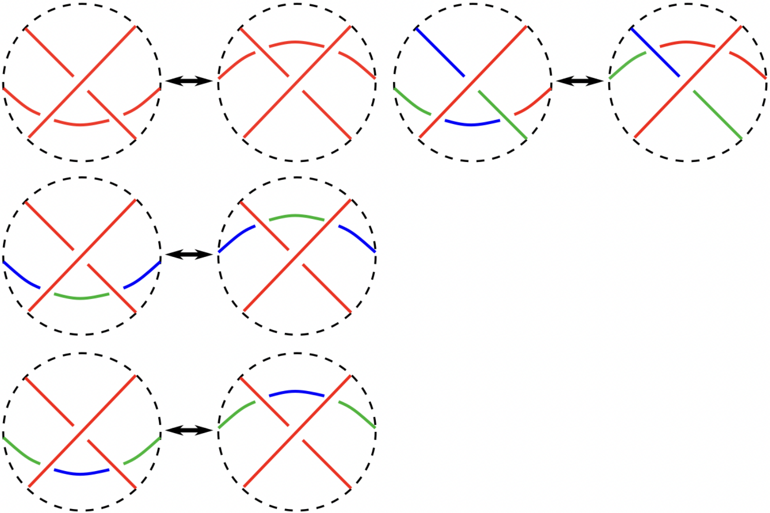

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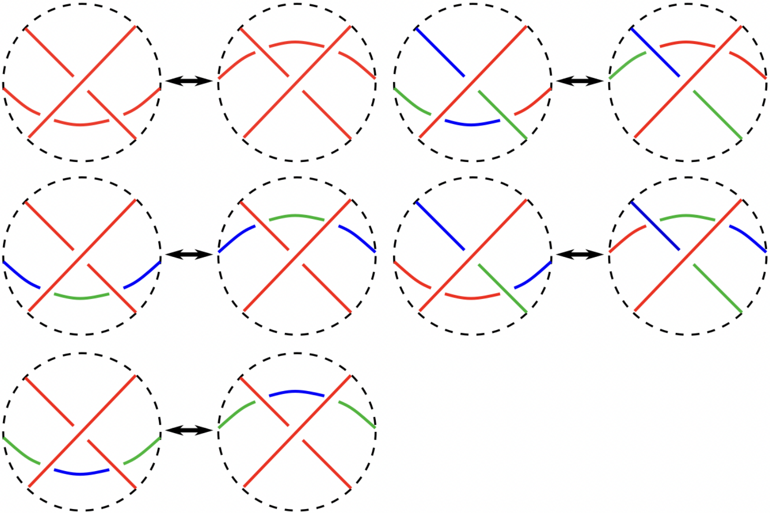

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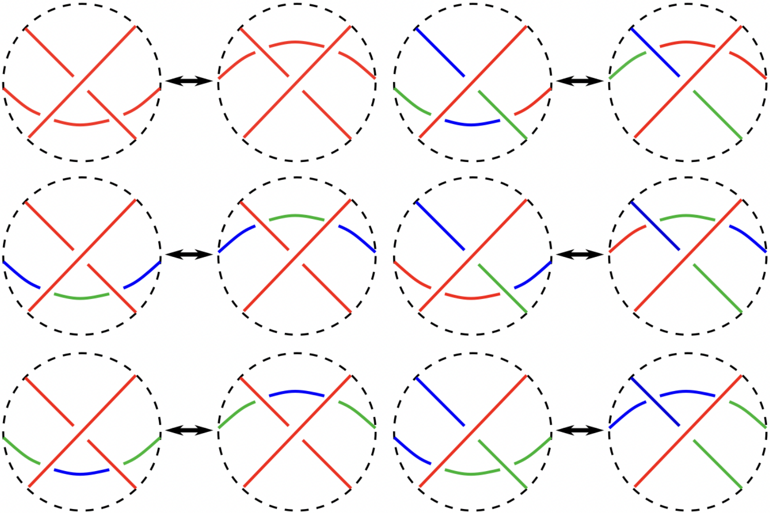

Type II Invariant Example

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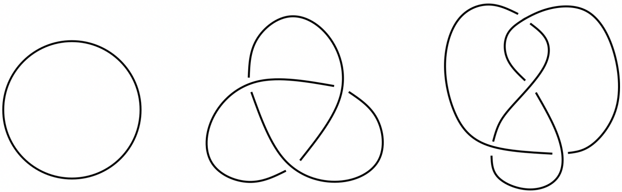
The **crossing number** of a knot K is the minimum number of crossings in any diagram of K.



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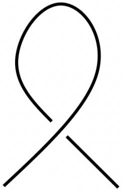

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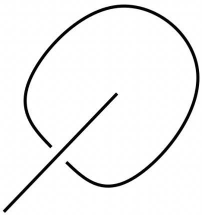

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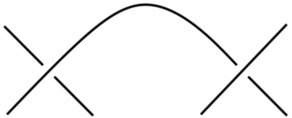

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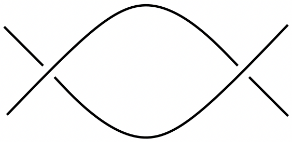

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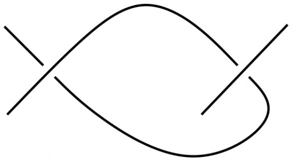

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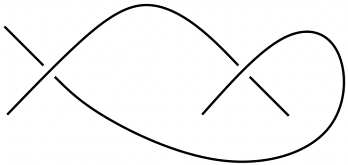

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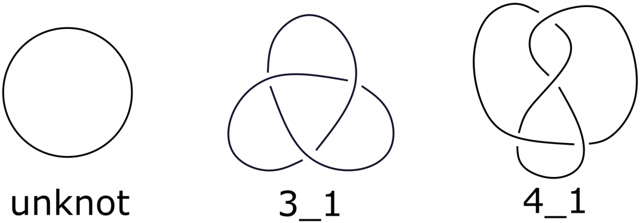

Type II Invariant Example

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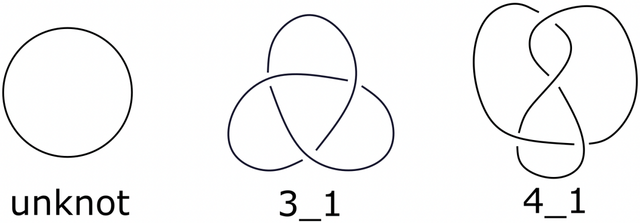
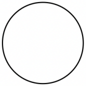
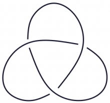
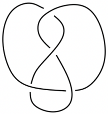
The **unknotting number** of a knot K is the minimum number of crossing changes required to transform K into the unknot.

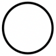


Our Invariants So Far

	 unknot	 3_1	 4_1
Tricolorable?	No	Yes	No
Crossing Number	0	3	4
Unknotting Number	0	1	1

Type I Invariant: Alexander Polynomial

Each knot K has an assigned Alexander polynomial $\triangle_K(t)$.

K	$\triangle_K(t)$
	1
3_1 	$t - 1 + t^{-1}$
4_1 	$t - 3 + t^{-1}$
5_1 	$t^2 - t + 1 - t^{-1} + t^{-2}$
5_2 	$2t - 3 + 2t^{-1}$

Type I Invariant: Alexander Polynomial

The set of Alexander polynomials are the Laurent polynomials (polynomials where powers of t can be negative) with integer coefficients where

- $\triangle_K(1) = \pm 1$ and
- $\triangle_K(t^{-1}) = \triangle_K(t)$

K	$\triangle_K(t)$
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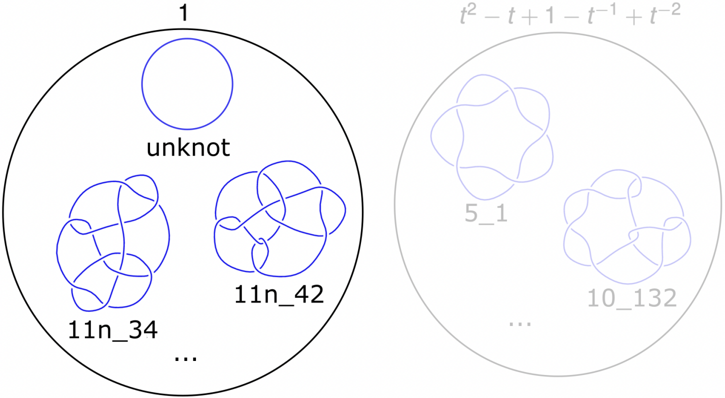
There are infinitely many knots realizing each Alexander polynomial.



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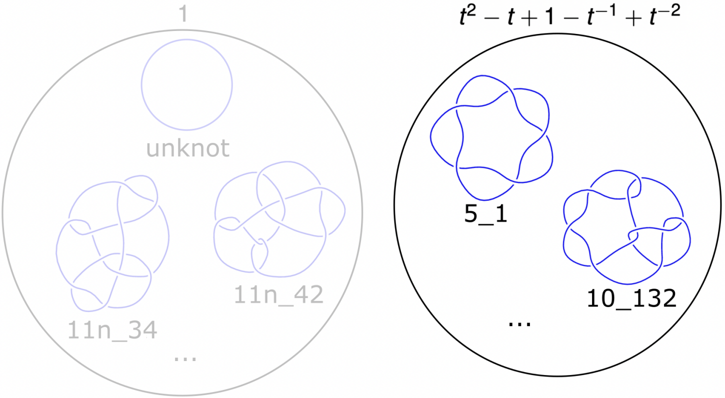

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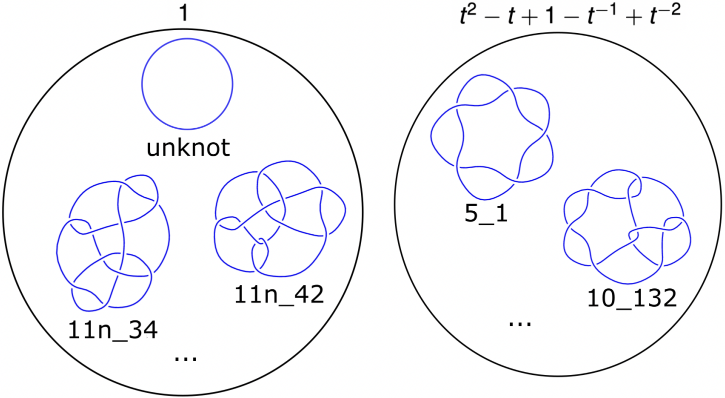

Alexander Polynomial and Crossing Changes

Theorem (Kondo, 1978)

For any Alexander polynomial $p(t)$, there exists a knot K with unknotting number one such that $\triangle_K(t) = p(t)$.

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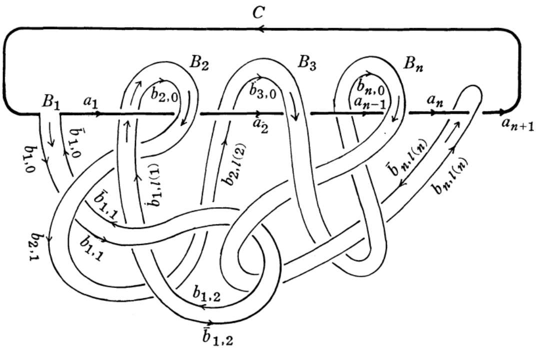

Fig. 4

Alexander polynomials and crossing changes

Definition

A **complete Alexander neighbor** is a knot K such that every possible Alexander polynomial is realized by a knot K' one crossing change away from K.

Question: Does there exist a complete Alexander neighbor with nontrivial Alexander polynomial?

Answer: I don't know yet! However, there are ways to narrow down the list of possible knots.

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Complete Alexander Neighbor

First, if a knot K has algebraic unknotting number greater than one, K is not a complete Alexander neighbor.

Theorem (A. W.)

Let K be a knot with unknotting number 1, where $|\triangle_K(-1)| \geq 3$ and where $|\triangle_K(-1)|$ is composite or $|\triangle_K(-1)| \equiv 1 \mod 4$. Then K is not a complete Alexander neighbor.

Corollary (A. W.)

Let K be a knot with a breadth 2 Alexander polynomial $\triangle_K(t) = n(t + t^{-1}) + 1 - 2n$. If K has unknotting number one or $1 - 4n$ is not a square, then K is not a complete Alexander neighbor.

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Unknotting Number

Theorem (Nakanishi & Okada, 2012)

Let K and K' be knots one crossing change apart. If K has unknotting number 1, then $|\triangle_{K'}(-1)| \equiv \pm n^2 \mod |\triangle_K(-1)|$ for some integer n.

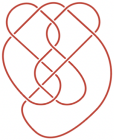
Theorem (A. W.)

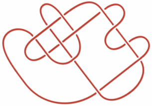
The five knots below have unknotting number greater than one.


11n_162


12n_805


12n_814


12n_844


12n_856

Unknotting Number

Theorem (Nakanishi & Okada, 2012)

Let K and K' be knots one crossing change apart. If K has unknotting number 1, then $|\triangle_{K'}(-1)| \equiv \pm n^2 \mod |\triangle_K(-1)|$ for some integer n.

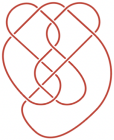
Theorem (A. W.)

The five knots below have unknotting number greater than one.

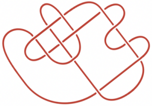
11n_162

12n_805

12n_814



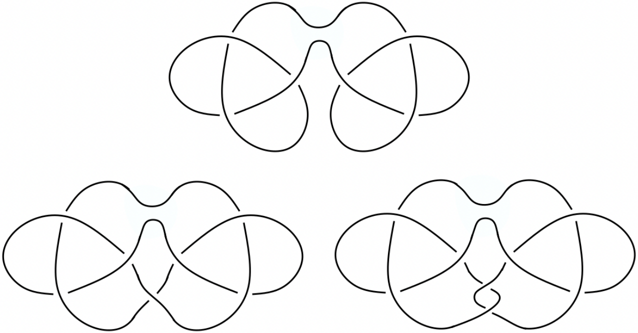
12n_844



12n_856

Polymath Jr. Project

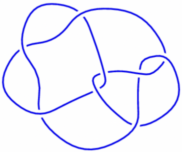
A **symmetric union presentation** of a knot K is a diagram of K built from a smaller knot (called a **partial knot** of K) joined with its mirror image.



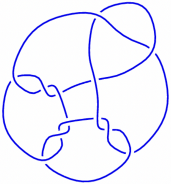
Polymath Jr. Project

Theorem (Ben Clingenpeel, Zongzheng (Jason) Dai, Gabriel Diraviam, Kareem Jaber, Ziyun Liu, Teo Miklethun, Haritha N, Michael Perry, Moses Samuelson-Lynn, Eli Seamans, Krishnendu Kar, Nicole Xie, Ruiqi Zou, A. W., Alex Zupan)

There exist knots K_1 and K_2 such that $|\triangle_{K_1}(-1)| = |\triangle_{K_2}(-1)|$, but they are not both partial knots of any knot K.



8_10



12n_642